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# LIE ALGEBRAICAL ASPECTS OF THE QUANTUM STATISTICS. UNITARY QUANTIZATION (A-QUANTIZATION)

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It is shown that the second quantization axioms can, in principle, be satisfied with creation and annihilation operators generating (in the case of $n$ pairs of such operators) the Lie algebra $A_n$ of the group $SL(n+1)$. A concept of the Fock space is introduced. The matrix elements of these operators are found.

## 0. Foreword

This manuscript was published as a JINR Preprint E17-10550 in 1977. It was accepted for publication in *Comm. Math. Phys.* under the condition to be shortened. Since I never did this, it remained unpublished. In view of the recent interest in various new kinds of statistics it seems to me the results bellow may be still of some interest. I publish them without any changes, although one could have added now a lot of new references. I apologize also for the somewhat old fashioned exposition.

## 1. Introduction

In the present paper we study some of the possible generalizations of the quantum statistics and more precisely of the second quantization procedure from a Lie algebraical point of view. The consideration is made in the frame of the Lagrangian field theory, however the results can be easily extended to other cases, e.g., to nuclear or solid state physics.

As is known [1], the ordinary quantum statistics can be considerably generalized if one quantizes the fields according to a weaker system of axioms, abandoning the usually accepted $C$-number postulate, i.e., the requirement for the commutator or the anticommutator of two fields to be a $C$-number. In this case the anticommutation relations between the Fermi creation and annihilation operators $f_i^+$ and $f_i^-$ [b)]

$$\{f_i^\xi, f_j^\eta\} = \frac{1}{4}(\xi-\eta)^2\delta_{ij} \tag{1}$$

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[b)] Throughout the paper the indices $\xi, \eta, \epsilon, \delta$ take values $\pm$ or $\pm 1$, $\{x,y\} = xy + yx$, $[x,y] = xy - yx$.

can be replaced by a weaker system of double commutation relations for the so-called para-Fermi operators $b_i^{\pm}$, namely

$$[[b_i^\xi, b_j^\eta], b_k^\epsilon] = \frac{1}{2}(\eta - \epsilon)^2 \delta_{jk} b_i^\xi - \frac{1}{2}(\xi - \epsilon)^2 \delta_{ik} b_j^\eta. \tag{2}$$

The commutation relations (2) exhibit some remarkable Lie algebraical properties. It turns out that the para-Fermi operators generate the algebra of the orthogonal group [2]. To make the statement more precise, consider a finite number of operators $b_1^{\pm}, \ldots, b_n^{\pm}$. Then the linear envelope over $\mathbf{C}$ of the operators

$$b_i^\xi,\ [b_j^\eta, b_k^\epsilon], \quad i, j, k = 1, \ldots, n \tag{3}$$

is isomorphic to the classical Lie algebra $B_n$ of the orthogonal group $SO(2n+1)$ [3].

There exists one-to-one correspondence between the representations of $B_n$ and the representations of $n$ pairs of para-Fermi operators [4]. Therefore the para-Fermi quantization is actually a quantization according to representations of the algebra of the orthogonal group in odd dimension and therefore may be called an odd-orthogonal quantization. This is an important point, a first hint that the group theory can in principle be relevant for the quantum statistics.

The algebras $B_n,\ n = 1, 2, \ldots$ constitute one of the four infinite series of the so-called classical Lie algebras. In the Cartan notation (which we follow) they are denoted as $A_n,\ B_n,\ C_n$ and $D_n$ for algebras of rank $n$, $n = 1, 2, \ldots$. The corresponding groups $SL(n),\ SO(2n+1),\ Sp(2n)$ and $SO(2n)$ are well known and therefore we do not define them here.

Once the Lie algebraic structure of the para-Fermi statistics is established, it is natural to ask whether one can quantize according to representations of the other classical Lie algebras. In the present paper we consider this question in connection with the algebra of the unimodular group.

In Sect. 3 we determine the concept of $A-$statistics, i.e., statistics with creation and annihilation operators ($a-$operators) that generate the algebra of the unimodular group. Next (Sect. 4) we define the Fock spaces $W_p,\ p = 1, 2, \ldots$ and the selection rules for the $A$-statistics. The integer $p$, called the order of the statistics, has well defined physical meaning: this is the maximal number of particles that can exist simultaneously (lemma 4). Tn Sect. 5 we calculate the matrix elements of the $a-$operators. In the limit $p \to \infty$ the $a-$operators reduce (up to a constant) to Bose operators.

The mathematics used in the paper is mainly of Lie algebraical nature. In order to introduce the notation and to make the exposition reasonably self-consistent, we collect in the next section some definitions and properties from the Lie algebra theory.

## 2. Preliminaries and notations

Let $A$ be a semi-simple complex Lie algebra of rank $n$, $\mathcal{H}$ - its Cartan subalgebra. By $\omega_i,\ e_{\omega_i},\ i = 1, 2, \ldots, p$ we denote the roots and the root vectors of $A$. The roots $\omega_i$ are vectors from the conjugate space $\mathcal{H}^*$ of $\mathcal{H}$. Sometimes it is convenient to consider them as vectors from $\mathcal{H}$ using the fact that every linear functional $\lambda^* \in \mathcal{H}^*$ can be uniquely represented in the form

$$\lambda^*(h) = (h, \lambda), \quad \forall\ h \in \mathcal{H}. \tag{4}$$

Here ( , ) is the Cartan-Killing form on $A$ and $\lambda \in \mathcal{H}$. The mapping

$$\theta : \ \lambda^* \to \lambda \equiv \theta\lambda^* \tag{5}$$

of $\mathcal{H}^*$ on $\mathcal{H}$ is one-to-one. From now on we consider the roots or any other linear functionals either as elements from $\mathcal{H}^*$ or from $\mathcal{H}$, denoting them in both cases by the same symbol (i.e., for $\lambda^*$ we write also $\lambda$).

With this agreement we can write

$$[h, e_{\omega_i}] = \omega_i(h)e_{\omega_i} = (h, \omega_i)e_{\omega_i} \quad \forall h \in \mathcal{H}. \tag{6}$$

The Cartan-Killing form defines a scalar product in the space $\mathcal{H}^r$ which is the real linear envelope of all roots; $\mathcal{H} = \mathcal{H}^r + \mathrm{i}\mathcal{H}^r$. Let $h_1, \ldots, h_n$ be an arbitrary covariant basis in $\mathcal{H}^r$ (and hence a basis in $\mathcal{H}$). The root $\omega_i$ is said to be positive (negative) if its first non-zero coordinate is positive (negative). The simple roots, i.e., those positive roots which cannot be represented as a sum of other positive roots, constitute a basis in $\mathcal{H}$. Any positive (negative) root is a linear combination of simple roots with positive (negative) integer coefficients.

Consider an arbitrary finite-dimensional irreducible $A-$module $W$ (i.e., a space where a finite dimensional irreducible representation of $A$ is realized. The basis $x_1, \ldots, x_N$ in $W$ can always be chosen such that

$$hx_i = \lambda_i(h)x_i = (h, \lambda_i)x_i \quad \forall\, h \in \mathcal{H},\ i = 1, \ldots, N. \tag{7}$$

Thus, to every basic vector $x_i \in W$ there corresponds an image $\lambda \in \mathcal{H}^*$ or $\mathcal{H}$. The vectors $x_i$ are the weight vectors and their images - the weights of the $A-$module $W$. The mapping $\tau : \ x_i \to \lambda_i$ is surjective and the number of the vectors $\tau^{-1}(\lambda_i)$ is called multiplicity of the weight $\lambda_i$. Let $e_\omega$ be a root vector and $\lambda_i$ be the weight of $x_i$. Then $e_\omega x_i$ is either zero or a weight vector with weight $\omega + \lambda_i$. The $A-$module $W$ contains a unique (up to multiplicative constant) weight vector $x_\Lambda$ with properties $e_{\omega_i} x_\Lambda = 0$ for all positive roots $\omega_i > 0$. The weight $\Lambda$ of $x_\Lambda$ is the highest weight of $W$. The multiplicity of $\Lambda$ is one and $W$ is spanned over all vectors

$$e_{\omega_{i_1}} e_{\omega_{i_2}} \ldots e_{\omega_{i_m}} x_\Lambda \quad m = 1, 2, \ldots, \tag{8}$$

where $\omega_{i_1}, \ldots, \omega_{i_m}$ are negative roots. Therefore an arbitrary weight $\Lambda$ is of the form

$$\lambda = \Lambda - \sum_{\omega_i > 0} k_i \omega_i \tag{9}$$

with $k_i$ positive integers and sum over positive (or only simple) roots.

Let $\pi_1, \ldots, \pi_n$ be the simple roots of $A$. Then for an arbitrary weight $\lambda$ the $n-$tuple $[\lambda_1, \lambda_2, \ldots, \lambda_n]$ has integer coordinates defined as

$$\lambda_i = \frac{2(\lambda, \pi_i)}{(\pi_i, \pi_i)} \quad i = 1, 2, \ldots, n. \tag{10}$$

The $n-$ tuple $[\Lambda_1, \ldots, \Lambda_n]$ corresponding to the highest weight $\Lambda$ has non-negative coordinates, and it defines the irreducible representation of $A$ in $W$ up to equivalence. On the contrary, to every vector $\Lambda \in \mathcal{H}$, such that $\Lambda_1, \ldots, \Lambda_n$ defined from (10) are non-negative integers, there corresponds an irreducible $A-$module. Thus there exists a one-to-one correspondence between the irreducible (finite-dimensional) $A-$modules and the set $[\Lambda_1, \ldots, \Lambda_n]$ of non-negative integers. We call $[\lambda_1, \ldots, \lambda_n]$ canonical co-ordinates of $\lambda$.

Define an $F-$basis $f_1, f_2, \ldots, f_n$ in $\mathcal{H}$ (or in $\mathcal{H}^*$) as follows

$$f_i = \frac{2}{(\pi_i, \pi_i)}\pi_i, \quad i = 1, \ldots, n \tag{11}$$

and let $K = \{f^1, f^2, \ldots, f^n\}$ be the corresponding dual basis, i.e., $f^i(f_j) = (f^i, f_j) = \delta_{ij}$. For an arbitrary $\lambda \in \mathcal{H}^*$ we have

$$\lambda = \sum_i \lambda(f_i) f^i = \sum_i \frac{2(\lambda, \pi_i)}{(\pi_i, \pi_i)} f^i \tag{12}$$

and therefore in the $K-$basis the coordinates of every weight $\lambda$ coincide with its canonical co-ordinates.

By means of the $F-$basis one can easily calculate the canonical co-ordinates of an arbitrary weight $\lambda$. Indeed

$$f_i x_\lambda = \lambda(f_i) x_\lambda = \frac{2(\lambda, \pi_i)}{(\pi_i, \pi_i)} x_\lambda \tag{12$'$}$$

and therefore the $i^{th}$ canonical co-ordinate $\lambda_i$ of $\lambda$ is an eigenvalue of $f_i$ on $x_\lambda$. More generally, if $h_1, \ldots, h_n$ is any covariant basis in $\mathcal{H}$, then the covariant co-ordinates $\lambda_1, \ldots, \lambda_n$ of the weight $\lambda$, i.e., the co-ordinates of $\lambda$ in the dual (or contravariant) basis $h^1, \ldots, h^n$ are determined from the relation

$$h_i x_\lambda = \lambda(h_i) x_\lambda = \lambda_i x_\lambda. \tag{13}$$

An important property of the set $\Gamma$ of all weights is its invariance under the Weyl group $S$, which is a group of transformations of $\mathcal{H}$. $S = \{S_{\omega_i} | \omega_i \; - \; roots \; of \; \Lambda\}$ is a finite group, its elements labelled by the roots $\omega_i$ of $A$ are defined as follows:

$$S_{\omega_i}.h = h - \frac{2(h, \omega_i)}{(\omega_i, \omega_i)}\omega_i \quad \forall \; h \in \mathcal{H}. \tag{14}$$

The set $\Gamma$ of all weights is characterized by the following statement: if $\lambda \in \Gamma$, then

$$S_{\omega_i}\lambda = \lambda + j\omega_i \in \Gamma, \quad j \; - \; integer. \tag{15}$$

and $\Gamma$ contains also the weights

$$\lambda, \lambda + \omega_i, \lambda + 2\omega_i, \ldots, \lambda + j\omega_i. \tag{16}$$

All weights that can be connected by transformations of the Weyl group are called equivalent. They have the same multiplicity. Among the equivalent weights there exists only one weight, the dominant one, the canonical co-ordinates of which are nonnegative integers.

## 3. Unitary quantization ($A-$quantization)

In the case of ordinary statistics the second quantization in the Lagrangian field theory can be performed in different equivalent ways. One can start, for instance, from the equal-time commutation relations. For the generalizations we wish to consider, it is more convenient to follow the quantization procedure accepted by Bogoljubov and Shirkov [6].

Apart from the fact that the fields become operators and the requirement for relativistic invariance, their approach is essentially based on what we call a main quantization postulate: the energy-momentum vector

$P^m$ and the angular-momentum tensor $M^{mn},\ m,n=0,1,2,3$ are expressed in terms of the operator-fields by the same expressions as in the classical case.

It follows from this postulate, together with the requirement that the field transforms according to unitary representations of the Poincare group and the compatibility of the transformation properties of the field and the state vectors, that the field $\Psi(x)$ satisfies the commutation relations

$$[P^m, \Psi(x)] = -i\partial^m \Psi(x). \tag{17}$$

This relation expresses (in infinitesimal form) the translation invariance of the theory.

To proceed further, it is convenient to pass to the discrete notation in the momentum space. Consider the field $\Psi(x)$ with a mass $m$ locked in a cube with edge $L$. For the eigenvalues $k_n^m$ of the 4-momentum $P^m,\ m=0,1,2,3$, one obtains

$$k_n^\alpha = \frac{2\pi}{L} n^\alpha, \quad k_n^0 = \sqrt{m^2 + (\frac{2\pi}{L})^2[(n^1)^2 + (n^2)^2 + (n^3)^2]}, \tag{18}$$

where $n=(n^1,n^2,n^3),\ \alpha=1,2,3$ and $n^\alpha$ runs over all non-negative integers. In momentum space the relation (17) reads as follows

$$[P^m, a_i^\pm] = \pm k_i^m a_i^\pm, \tag{19}$$

where $a_i^+$ ($a_i^-$) are the corresponding to $\Psi(x)$ creation and annihilation operators and the index $i$ replace all discrete indices ($n$, spin, charge, etc.).

The commutation relations between the creation and annihilation operators are usually derived from the translation invariance law in momentum space (19). We call it *initial quantization equation* (IQE). To determine the commutation relations one has to specify one more point. Up to now nothing was said about the creation and the annihilation operators that enter into $P^m$. In the ordinary theory it is usually accepted that the dynamical variables are written in a normal-product form and therefore for the Fermi fields this gives

$$P^m = \sum_i k_i^m f_i^+ f_i^-, \tag{20}$$

where $f_i^+$ ($f_i^-$) are the Fermi creation (annihilation) operators (1). One can easily check that the initial quantization equation (with $a_i^\pm = f_i^\pm$) is compatible with the anticommutation relations (1). This is, however, not the case for the para-Fermi operators (2), apart from the case of their Fermi representation. The para-Fermi statistics cannot be derived from the normal-product form of the dynamical variables. In order to fulfil (19) Green chose another ordering of the operators in $P^m$ and in particular for spinor fields he wrote [1]:

$$P^m = \frac{1}{2}\sum_i k_i^m [b_i^+, b_i^-]. \tag{21}$$

We see that the ordering of the operators in the 4-momentum is closely related to the corresponding statistics. It is natural to expect therefore that any other generalization of the statistics may require new expressions for $P^m$. In order to get a feeling as to how one can modify $P^m$, we now proceed to derive the para-Fermi statistics in such a way that later on it will be possible to generalize the idea to other case.

We start with the expression (20). In order to use a proper Lie algeraical language (finite-dimensional Lie algebras), suppose that the sum in (20) is finite,

$$P^m = \frac{1}{2}\sum_{i=1}^{n} k_i^m [b_i^+, b_i^-]. \tag{22}$$

This is only an intermediate step. In the final results we let $n \to \infty$.

As we have already mentioned, the set $f_1^\pm, \ldots, f_n^\pm$ of Fermi creation and annihilation operators (1) generates on particular representation (we call it *Fermi representation*) of the algebra $B_n$. We put now the question: can the expression (22) be written in such a form that the initial quantization equation (19) will hold for the Fermi operators, considered as generators of $B_n$, i.e., independently of the fact that we are staying in one particular representation of $B_n$ - the Fermi one. The Lie algebraical reason why (19) does not hold for the para-Fermi operators is clear. It is due to the fact that the 4-momentum (22) does not belong to $B_n$ since it contains products of $b_i^+$ and $b_i^-$, which is not a Lie algebraical operation. Therefore the IQR, considered as commutation relation, is not preserved for other representations of $B_n$. If however the 4-momentum together with the creation and annihilation can be embedded in a Lie algebra, so that in the Fermi case $P^m$ reduces to (22), then the IQR (19) will hold for any other representation of this algebra.

For this purpose we rewrite the 4-momentum (22) in the following identical form

$$P^m = \sum_{i=1}^{n} k_i^m (\frac{1}{2}[f_i^+, f_i^-] + \frac{1}{2}\{f_i^+, f_i^-\}). \tag{23}$$

Consider the Lie algebra generated from $f_1^\pm, \ldots, f_n^\pm$ and $\{f_i^+, f_i^-\}$. Since $\{f_i^+, f_i^-\} = 1$, we obtain the algebra $B_n \oplus I$, where $I$ is the one-dimensional commutative center. Now $P^m \in B_n \oplus I$ and therefore the commutation relation (19) holds for any other representation. In other words, if we substitute in (23) $f_i^\pm \to b_i^\pm$ and $\{f_i^+, f_i^-\} \to \mathbf{1}$, i.e., put

$$P^m = \sum_{i=1}^{n} k_i^m (\frac{1}{2}[b_i^+, b_i^-] + \frac{1}{2}\mathbf{1}), \tag{24}$$

where $\mathbf{1}$ is the generator of the center of $B_n \oplus I$, then the initial quantization condition (19) will be fulfilled for any representation of $B_n \oplus I$.

The operator

$$Q^m = \sum_{i=1}^{n} k_i^m \frac{1}{2}\mathbf{1} \tag{25}$$

commutes with all creation and annihilation operators and all elements of $B_n \oplus I$. Therefore it is a constant within every irreducible representation and in the particular case of para-Fermi statistics the second term in (24) can be omitted. Thus, we obtain the expression (21) for $P^m$, postulated by Green from the very beginning.

We shall now apply a similar approach for the algebra $A_n$ of the unimodular group $SL(n+1)$. The nontrivial part is to find an analogue of the Fermi operators, i.e., operators $a_i^\pm$ that generate some representation of $A_n$ and fulfil the initial quantization equation (19) with 4-momentum written (in this particular representation) in a normal product form. Then we shall apply the above procedure to enlarge the class od admissible representations.

First we recall some properties of $A_n$. We consider $A_n$ as a subalgebra of the algebra $gl(n+1)$ of the general linear group $GL(n+1)$. The algebra $gl(n+1)$ may be determined as a linear envelope of the generators $e_{ij},\ i,j=0,1,\ldots,n$, that satisfy the commutation relations

$$[e_{ij},e_{kl}]=\delta_{jk}e_{il}-\delta_{li}e_{kj},\quad i,j,k,l=0,1,\ldots,n. \tag{26}$$

Let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be the Cartan subalgebras of $A_n$ and $gl(n+1)$, resp. Denote by $env\{X\}$ the linear envelope of an arbitrary set $X$. In terms of the $gl(n+1)$ generators we have:

$$\begin{aligned}
&gl(n+1)=env\{e_{ij}|i,j=0,1,\ldots,n\},\\
&A_n=env\{e_{ii}-e_{jj},\ e_{ij}|i\neq j=0,1,\ldots,n\},\\
&\tilde{\mathcal{H}}=\{h_i|h_i=e_{ii},\ i=0,1,\ldots,n\},\\
&\mathcal{H}=\{h_i-h_j|h_i=e_{ii},\ i,j=0,1,\ldots,n\}.
\end{aligned} \tag{27}$$

For a covariant basis in $\tilde{\mathcal{H}}$ we choose the vectors ($h_i\equiv e_{ii}$)

$$h_0,h_1,\ldots,h_n. \tag{28}$$

The algebra $gl(n+1)$ is not semi-simple. Its Killing form is degenerate and does not determine a scalar product on $\tilde{\mathcal{H}}^r$. It is convenient to introduce a metric in $\tilde{\mathcal{H}}$ with the relation

$$(h_i,h_j)=2(n+1)\delta_{ij}. \tag{29}$$

Restricted on $\mathcal{H}$ this metric coincides with the Cartan-Killing form on $A_n$.

From (26) and (29) one obtains

$$[h,e_{ij}]=(h,h^i-h^j)e_{ij}\quad \forall h\in\mathcal{H},\ i\neq j=0,1,\ldots,n, \tag{30}$$

where $h^0,h^1,\ldots,h^n$ is the contravariant (i.e., the dual to $h_0,h_1,\ldots,h_n$) basis in $\tilde{\mathcal{H}}$. Hence the generators $e_{ij},\ i\neq j=0,1,\ldots,n$ are the root vectors of $A_n$. The correspondence with their roots is

$$e_{ij}\rightarrow h^i-h^j,\quad i\neq j=0,1,\ldots,n. \tag{31}$$

In the basis (28) the generators

$$e_{ij},\quad i<j\ (i>j),\quad i,j=0,1,\ldots,n \tag{32}$$

are the positive (negative) root vectors of $A_n$. The simple roots are

$$\pi_i=h^{i-1}-h^i,\quad i=1,\ldots,n. \tag{33}$$

Therefore the $F$-basis (11) in this case reads as

$$f_i=\frac{2}{(\pi_i,\pi_i)}\pi_i=h_{i-1}-h_i,\quad i=1,\ldots,n \tag{34}$$

We are now ready to define the analogue of the Fermi operators. Let $E_{ij},\ i,j = 0,1,\ldots,n$ be $(n+1)-$square matrix with 1 on the intersection of $i-$row and $j-$column and zero elsewhere. Clearly the mapping

$$\pi:\ e_{ij} \to E_{ij}, \quad i,j = 0,1,\ldots,n \tag{35}$$

determines a representation of $gl(n+1)$ and hence its restriction on $A_n$ gives a representation of $A_n$.

The operators

$$A_i^+ = E_{i0},\ \ A_i^- = E_{0i}, \quad i = 1,2,\ldots,n \tag{36}$$

generate the algebra $A_n$ (in the above representation) since

$$[A_i^+, A_j^-] = E_{ij}, \quad [A_k^+, A_k^-] = E_{kk} - E_{00}, \quad i \neq j, \quad i,j,k = 1,\ldots,n. \tag{37}$$

Moreover for the commutation relations between $A_1^\pm,\ldots,A_n^\pm$ and the operator

$$P^m = \sum_i k_i^m A_i^+ A_i^- \tag{39}$$

we obtain the right expression:

$$[P^m, A_i^\pm] = \pm k_i^m A_i^\pm. \tag{39}$$

The operators $A_1^\xi,\ldots,A_n^\xi$ satisfy the initial quantization equation and can be considered as creation $(\xi = +)$ and annihilation $(\xi = -)$ operators.

The commutation relation (39) does not hold for other representations of $A_n$. In order to extend the class of the admissible representations, we represent the 4-momentum (38) like in the Fermi case, in the form

$$P^m = \sum_i k_i^m([A_i^+, A_i^-] + E_{00}). \tag{40}$$

Consider now the Lie algebra generated from the operators $A_1^\pm,\ldots,A_n^\pm$ and $E_{00}$. One can easily show, it is the algebra $gl(n+1) = A_n \oplus I$. Since $P^m \in gl(n+1)$, the initial quantization equation (39) holds for any other representation of $gl(n+1)$. Hence we may define representation independent creation and annihilation operators as follows

$$a_i^+ = e_{i0},\ \ a_i^- = e_{0i}, \quad i = 1,2,\ldots,n. \tag{41}$$

In this case we have to postulate for $P^m$ the expression

$$P^m = \sum_i k_i^m([a_i^+ a_i^-] + e_{00}). \tag{42}$$

The operators $a_i^\pm$ are root vectors of $A_n$. The correspondence with their roots is

$$a_i^\pm \ \leftrightarrow\ \mp(h^0 - h^i), \quad i = 1,\ldots,n, \tag{43}$$

and therefore the creation (annihilation) operators are negative (positive) root vectors. Since any other root $h^i - h^j,\ i \neq j = 1,\ldots,n,$

$$h^i - h^j = (h^0 - h^j) - (h^0 - h^i)$$

is a sum of the roots of $a_j^-$ and $a_i^+$, the creation and the annihilation operators generate the algebra $A_n$.

The commutation relations of $A_n$ can be written in terms of $a_i^{\pm}$ only. From (26) we obtain

$$
\begin{aligned}
&[[a_i^+, a_j^-], a_k^+] = \delta_{kj} a_i^+ + \delta_{ij} a_k^+, \\
&[[a_i^+, a_j^-], a_k^-] = -\delta_{ki} a_j^- - \delta_{ij} a_k^-, \\
&[a_i^+, a_j^+] = [a_i^-, a_j^-] = 0.
\end{aligned} \tag{44}
$$

*Definition 1.* The operators $a_i^{\pm},\ i = 1, 2, \ldots$ satisfying the commutation relations (44) are called $a-$*operators* and the corresponding quantization (statistics) *unitary or* $A-$*quantization (statistics).*

We observe that the equal-frequency operators commute with each other. This property helps a lot in all calculations with the $a-$operators.

## 4. Fock spaces for the a-operators

We now proceed to study those representations of the $a-$creation and annihilation operators that possess the main features of the Fock space representations in the ordinary quantum mechanics. We continue to consider a finite set of operators. The extension of the results to the infinite (including continuum) number of $a-$operators will be evident.

*Definition 2.* Let $a_1^{\xi}, \ldots, a_n^{\xi}$ be $a-$creation ($\xi = +$) and annihilation ($\xi = -$) operators. The $A_n-$module $W$ is said to be a Fock space of the algebra $A_n$ if it fulfills the conditions:

1. *Hermiticity condition*

$$
(a_i^+)^* = a_i^-, \quad i = 1, \ldots, n. \tag{45}
$$

Here $^*$ denotes hermitian conjugation operation.

2. *Existence of vacuum.* There exists a vacuum vector $|0\rangle \in W$ such that

$$
a_i^- |0\rangle = 0, \quad i = 1, \ldots, n. \qquad (46).
$$

3. *Irreducibility.* The representation space $W$ is spanned over all vectors

$$
a_{i_1}^+ a_{i_2}^+ \ldots a_{i_m}^+ |0\rangle, \quad m \in N_0. \tag{47}
$$

By $N_0$ we denote the set of all non-negative integers. The Fock space of $A_n$ is called also $A_n-$module of Fock, Fock module of the $a-$operators or simply Fock module.

*Lemma 1.* The hermiticity condition (45) can be satisfied if and only if the $A_n-$module $W$ is a direct sum of finite-dimensional modules.

*Proof.* The generators of the compact from $su(n+1)$ of $A_n$ read in terms of the $a-$operators as follows

$$
\begin{aligned}
&E_{0j} = i(a_j^+ + a_j^-), \\
&F_{0j} = a_j^- - a_j^+, \\
&E_{jk} = i[a_j^+, a_k^-] + i[a_k^+, a_j^-], \\
&F_{jk} = [a_j^+, a_k^-] - [a_k^+, a_j^-].
\end{aligned} \tag{48}
$$

Evidently the generators are antihermitian if and only if (45) holds.

As is known, the antihermitian representations of the compact forms of the classical algebras are completely reducible. The irreducible components are finite-dimensional. This proves the sufficient part. The necessity follows from the observation that the metric in any irreducible $su(n)$-module can be introduced so that the generators are antihermitian.

From the complete reducibility and the irreducibility condition (definition 1) we conclude.

*Corollary 1.* The Fock spaces are finite-dimensional irreducible $A_n-$modules.

In the remaining part of the paper by creation and annihilation operators we always mean $a-$operators. Moreover we fix the ordering of the basis in $\tilde{\mathcal{H}}$ to be (28). Then the creation and the annihilation operators $a_i^+$, $a_i^-$ are negative and positive root vectors. In this case the operators $a_1^-,\ldots,a_n^-$ annihilate the highest weight vector $x_\Lambda$ of the Fock space and hence $x_\Lambda$ is one of the candidates for a vacuum state.

*Lemma 2.* Let $W$ be a Fock space of $A_n$. Up to a multiplicative constant the vacuum state is unique and coincides with the highest weight vector $x_\Lambda$ of $W$

*Proof.* First suppose the vacuum is a weight vector $x_\lambda \neq x_\Lambda$. Then the corresponding weights are also different, $\lambda \neq \Lambda$. Moreover $\Lambda > \lambda$ (i.e., the vector $\Lambda - \lambda$ is positive). The irreducibility condition says there exists a polynomial $P(a_1^+,\ldots,a_n^+)$ of the creation operators such that

$$x_\Lambda = P(a_1^+,\ldots,a_n^+)x_\lambda. \tag{49}$$

Denote by $\omega_i$ the root of $a_i^+$. From (49) we have

$$\Lambda = \lambda + \sum_{i=1}^{n} k_i\omega_i, \quad k_i \in N_0.$$

This is, however, impossible, since $\Lambda - \lambda > 0$ and $\sum_{i=1}^n k_i\omega_i < 0$. We conclude that the vacuum cannot be a weight vector different from $x_\Lambda$.

More generally, suppose $|0\rangle \in W$ is a vacuum state different from $x_\Lambda$. An arbitrary vector $x \in W$ and in particular $|0\rangle$ can be represented uniquely as a sum of weight vectors $x_{\lambda_i}$ with different weights $\lambda_i$:

$$|0\rangle = \sum_{j=0}^{m} x_{\lambda_j}, \quad \lambda_i \neq \lambda_j \quad if \quad i \neq j. \tag{50}$$

The vectors $x_{\lambda_0},\ldots,x_{\lambda_m}$ are linearly independent. The nonzero of the vectors $a_i^- x_{\lambda_0},\ldots,a_i^- x_{\lambda_m}$ are also linearly independent, since they correspond to different weights. Hence

$$a_i^-|0\rangle = 0 \ \text{ implies } \ a_i^- x_{\lambda_j} = 0, \ \ j = 0,1,\ldots,m. \tag{51}$$

Let for definiteness $\lambda_0 > \lambda_1 > \ldots > \lambda_m$. The vector cannot be a vacuum state if $\lambda_0 \neq \Lambda$ since clearly there exists no polynomial $P(a_1^+,\ldots,a_n^+)$ such that

$$x_\Lambda = P(a_1^+,\ldots,a_n^+)|0\rangle. \tag{52}$$

Suppose $|0\rangle = x_\Lambda + x_{\lambda_1} + \ldots + x_{\lambda_m}$. Then (52) can be satisfied only when there exists a monomial $(a_1^+)^{l_1}\ldots(a_n^+)^{l_n}$ with the property

$$x_{\lambda_1} = (a_1^+)^{l_1}\ldots(a_n^+)^{l_n}x_\Lambda.$$

This is, however, impossible since for $l_i \neq 0$ $a_i^- x_{\lambda_1} \neq 0$ and this contradicts (51).

In the following theorem we prove one convenient criterion for the $A_n-$module to be a Fock space.

*Theorem 1.* The $A_n-$module $W$ is a Fock space if and only if it is an irreducible finite-dimensional module such that

$$a_i^- a_j^+ x_\Lambda = 0 \quad i \neq j = 1, \ldots, n. \tag{53}$$

The highest weight vector $x_\Lambda$ is the vacuum of $W$.

*Proof.* Let $W$ be a Fock space. Then it is finite-dimensional irreducible $A_n-$module (corollary 1) and the vacuum $|0\rangle = x_\Lambda$ (lemma 2). The operator $[a_i^-, a_j^+]$, $i \neq j$ is a root vector of $A_n$. Its root $h^j - h^i$ cannot be represented as linear combination of the roots $-h^0 + h^i$ of the creation operators $a_1^+, \ldots, a_n^+$. Hence there exists no polynomial $P(a_1^+, \ldots, a_n^+)$ of $a_1^+, \ldots, a_n^+$ such that

$$[a_i^- a_j^+]x_\Lambda = P(a_1^+, \ldots, a_n^+)x_\Lambda \neq 0$$

Since $a_i^- a_j^+ x_\Lambda \in W$ it has to be zero, $a_i^- a_j^+ x_\Lambda = 0$, $i \neq j$.

The proof of the sufficient part of the theorem is based on the Poincare-Birkhoff-Witt theorem [7]: Given a Lie algebra $A$ with a basis $e_1, \ldots, e_N$. All ordered monomials $e_1^{j_1} e_2^{j_2} \ldots e_N^{j_N}$ constitute a basis in the universal enveloping algebra $U$ of $A$.

Let in the irreducible finite-dimensional $A_n-$module $W$ the equality (53) holds. Divide the basis elements of $A_n$ into three groups

$$\begin{aligned}
&I = \{a_i^+,\ [a_j^+, a_k^+] |\ j < k;\ i,j,k = 1, \ldots, n\} \equiv \{e_{-1}, e_{-2}, \ldots, e_{-p}\},\\
&II = \{a_i^-,\ [a_j^-, a_k^-],\ [a_r^-, a_s^+] |\ j < k;\ r \neq s;\ i,j,k,r,s = 1, \ldots, n\} \equiv \{e_1, e_2, \ldots, e_q\},\\
&III = \{\omega_k | k = 1, \ldots, n\},
\end{aligned}$$

where $\omega_1, \ldots, \omega_n$ is a basis in the Cartan subalgebra $\mathcal{H}$. Order the elements within each group in an arbitrary way. From the irreducibility and the Poincare-Birkhoff-Witt theorem it follows that $W$ is linearly spanned on all vectors

$$e_{-1}^{i_1} \ldots, e_{-p}^{i_p} e_1^{j_1} \ldots e_q^{j_q} \omega_1^{k_1} \ldots \omega_n^{k_n} x_\Lambda. \tag{54}$$

Since $x_\Lambda$ is an eigenvector of the Cartan subalgebra and the operators from II annihilate $x_\Lambda$, the vector (54) is non-zero only if $j_1 = j_2 = \ldots = j_n = 0$. Hence $W$ is spanned on all vectors

$$P(a_1^+, \ldots, a_n^+)x_\Lambda,$$

where $P$ is an arbitrary polynomial of the creation operators. This proves that $W$ is a Fock space with a vacuum $|0\rangle = x_\Lambda$.

Now it remains to determine the irreducible $A_n-$modules satisfying the condition (53). In order to solve this problem, we consider first some questions from the representation theory of $A_n$. As we mentioned, it is

convenient to consider $A_n$ as a subalgebra of $gl(n+1)$. This possibility is based on the circumstance that the irreducible $gl(n+1)-$modules are also $A_n-$irreducible. On the other hand, every irreducible representation of $A_n$ in $W$ can be continued in infinitely many ways to an irreducible representation of $gl(n+1)$ in the same space. For this purpose it is enough to define the operator $f_0 = h_0 + h_1 + \ldots + h_n$ in $W$ where $h_0, h_1, \ldots, h_n$ is the covariant basis (28) in $\tilde{\mathcal{H}}$. Since $f_0$ commutes with $gl(n+1)$, $f_0$ has to be a constant in $W$, i.e.,

$$f_0 x = \Lambda_0 x \quad \forall x \in W \tag{55}$$

with $\Lambda_0$ being an arbitrary number. Let $f_1, \ldots, f_n$ be the $F-$ basis (34) in $\mathcal{H}$. Then

$$\tilde{F} = \{f_0, f_1, \ldots, f_n\} \tag{56}$$

defines a basis in the Cartan subalgebra $\tilde{\mathcal{H}} \subset gl(n+1)$.

The eigenvalues $\Lambda_0, \Lambda_1, \ldots, \Lambda_n$ of $\tilde{F}$ on the highest weight vector $x_\Lambda \in W$ characterize $W$ as an irreducible $gl(n+1)-$module. Let $x_{\lambda_1}, \ldots, x_{\lambda_N}$ be a basis of weight vectors in $W$. In view of (55) the $A_n$-weights $\lambda_1, \ldots, \lambda_N$ are naturally extended to linear functional on $\tilde{\mathcal{H}}$ from the requirement $\lambda_i(f_0) = \Lambda_0$. Then for any weight vector $x_\lambda$ we have

$$hx_\lambda = \lambda(h)x_\lambda = (h, \lambda)x_\lambda \quad h \in \tilde{\mathcal{H}}. \tag{57}$$

The numbers $\Lambda_0, \Lambda_1, \ldots, \Lambda_n$ are the co-ordinates of the highest weight $\Lambda$ in the basis

$$\tilde{K} = \{f^0, f^1, \ldots, f^n\} \tag{58}$$

dual to $\tilde{F}$. We call $\tilde{K}$ *a canonical basis* of $gl(n+1)$ and the co-ordinates $[\Lambda_0, \Lambda_1, \ldots, \Lambda_n]$ - *canonical co-ordinates* of the $gl(n+1)-$module $W$. The properties of the Weyl group, which we shall often use, read more simply in the orthogonal contravariant basis $h^0, h^1, \ldots, h^n$. From the equality

$$\Lambda = \sum_{i=0}^{n} \Lambda_i f^i = \sum_{i=0}^{n} L_i h^i$$

we obtain for the orthogonal co-ordinates $L_0, L_1, \ldots, L_n$ of the highest weight of $W$ the following expressions

$$\begin{aligned}
&L_0 = \frac{1}{n+1}[\Lambda_0 + n\Lambda_1 + (n-1)\Lambda_2 + \ldots + 1.\Lambda_n] \\
&L_1 = L_0 - \Lambda_1, \\
&L_2 = L_0 - \Lambda_1 - \Lambda_2, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&L_n = L_0 - \Lambda_1 - \Lambda_2 - \ldots - \Lambda_n.
\end{aligned} \tag{59}$$

Since in the $A_n-$module $W$ $\Lambda_1, \ldots, \Lambda_n$ are non-negative integers and $\Lambda_0$ is an arbitrary constant, it can be chosen such that all orthogonal co-ordinates $L_0, L_1, \ldots, L_n$ are integers. Moreover

$$L_0 \geq L_1 \geq L_2 \geq \ldots \geq L_n. \tag{60}$$

We pass now to the main problem of this section, classification of the Fock spaces. Unless otherwise stated, the roots and the weights are represented by their orthogonal co-ordinates in the contravariant orthogonal basis $h^0, h^1, \ldots, h^n$ in $\tilde{\mathcal{H}}$, i.e.,

$$\lambda = (l_0, l_1, \ldots, l_n) \equiv \sum_{i=0}^{n} l_i h^i. \tag{61}$$

*Theorem 2.* The irreducible $A_n-$module is a Fock space if and only if its highest weight is $(p, 0, \dots, 0)$; $p$ is an arbitrary positive integer.*

*Proof.* As we know (theorem 1), the Fock spaces are those and only those irreducible $A_n-$modules whose highest weight vectors $x_\Lambda$ are annihilated by all operators $a_i^- a_j^+$, $i \neq j = 1, \dots, n$, i.e.,

$$a_i^- a_j^+ x_\Lambda = 0 \quad i \neq j = 1, \dots, n. \tag{62}$$

Since $a_i^- x_\Lambda = 0$ and $[a_i^-, a_j^+ x_] = e_{ij}$ (62) can be replaced by the requirement

$$e_{ij} x_\Lambda = 0 \quad i \neq j = 1, \dots, n. \tag{63}$$

The generators $e_{ij}$ are root vectors of $A_n$ with roots (31), i.e.,

$$e_{ij} \leftrightarrow h^i - h^j, \quad i \neq j = 1, \dots, n.$$

For $i < j$ $e_{ij}$ is a positive root vector and (63) holds from the definition of $x_\Lambda$. It remains to determine those $A_n-$modules with weights

$$\Lambda = (L_0, L_1, \dots, L_n) \tag{64}$$

for which the sums

$$\Lambda + h^j - h^i, \quad i < j = 1, \dots, n \tag{65}$$

are not weights.

We shall use the properties (14) and (15) of the Weyl group $S$. According to (14) if $S_{h^i - h^j} \in S$ and $\lambda = (l_0, \dots, l_i, \dots, l_j, \dots, l_n)$ is a weight, then $S_{h^i-h^j}.\lambda$ is also a weight. Using the scalar product (29) we have

$$S_{h^i-h^j}.\lambda = \lambda - \frac{2(\lambda, h^i - h^j)}{(h^i - h^j, h^i - h^j)}(h^i - h^j) = (l_0, \dots, (l_j)_i, \dots, (l_i)_j, \dots, l_n) \tag{66}$$

where $(l_j)_i$ (resp. $(l_i)_j$) on the r.h.s. of (66) is to indicate that $l_j$ (resp. $l_i$) is situated on the place $i$ (resp. $j$), whereas any other $l_k$ is on the place $k$.

Thus, the Weyl group in this case reduces to (all possible) permutations of the orthogonal co-ordinates. For the highest weight (64) we have

$$S_{h^i-h^j}.\Lambda = (L_0, \dots, (L_j)_i, \dots, (L_i)_j, \dots, L_n) =$$
$$(L_0, \dots, L_i, \dots, L_j, \dots, L_n) + (L_i - L_j)(0, \dots, 0, (-1)_i, 0, \dots, 0, (1)_j, 0, \dots, 0).$$

According to (15) all vectors

$$(L_0, \dots, L_i, \dots, L_j, \dots, L_n) + k(0, \dots, 0, (-1)_i, 0, \dots, 0, (1)_j, 0, \dots, 0) \tag{67}$$

with $0 \leq k \leq L_i - L_j$ are also weight. As we know, for $i < j$ $L_i \geq L_j$. Suppose $L_i > L_j$. Then $k$ in (67) can be equal to one and

$$\lambda = \Lambda + h^j - h^i, \quad i < j$$

* the case $p = 0$ corresponds to the trivial one-dimensional representation.

is a weight. Hence the $A_n-$module $W$ is not a Fock space if in its orthogonal signature $\Lambda = (L_0, L_1, \ldots, L_n)$ there exists $L_i > L_j$ for $0 < i < j$.

It remains to consider the modules with

$$\Lambda = (L_0, L, \ldots, L), \quad L_0 \geq L. \tag{68}$$

Suppose for $0 < i < j$

$$\lambda = \Lambda + h^j - h^i = (L_0, L, \ldots, L, (L-1)_i, L, \ldots, L, (L+1)_j, L, \ldots, L)$$

is a weight. Then

$$\lambda' = (L_0, L, \ldots, L, (L+1)_i, L, \ldots, L, (L-1)_j, L, \ldots, L)$$

is also a weight. This is, however, impossible since $\lambda' > \Lambda$. Hence all $A_n-$modules with signatures (68) are Fock spaces.

We could have stopped the proof here since the signatures

$$(L_0, L, \ldots, L) \quad \text{and} \quad (L_0 - L, 0, \ldots, 0) \tag{69}$$

describe one and the same $A_n-$module. This could have been done if all information was carried by $A_n$, i.e., if the dynamical variables were functions of the generators of $A_n$ only. This is however not the case. The 4-momentum (42) $P^m \notin A_n$ although $P^m \in gl(n+1)$. Therefore physically the representations (69) are distinguishable.

We shall determine the orthogonal co-ordinates of $\Lambda$ from the requirement for the energy of the vacuum to be zero. In terms of the orthogonal basis (28) $P^m$ can be written as

$$P^m = \sum_{i=1}^{n} k_i^m h_i. \tag{70}$$

Since for $\Lambda = (L_0, L, \ldots, L) \quad h_i x_\Lambda = L x_\Lambda, \;\; i = 1, \ldots, n$, we require

$$P^m|0\rangle = \sum_{i=1}^{n} k_i^m L|0\rangle = 0, \quad m = 1, 2, 3. \tag{71}$$

Here $k_1^0, \ldots, k_n^0$ are analogue of the energy spectrum of the one-particle states, $k_i^0 > 0$ (see (18)). Therefore (71) implies $L = 0$.

Later on we shall see that $h_i, \; i = 1, \ldots, n$ is a number operator for particles in a state $i$. This together with (71) also gives $L = 0$.

Consider the Fock space $W_p$ with $\Lambda = (p, 0, \ldots, 0)$. Using the definition (41) of the $a-$operators, from (62) we have

$$a_i^- a_j^+ |0\rangle = p|0\rangle. \tag{72}$$

We obtain the same expression as in the case of parastatistics of order $p$ [8]. Therefore we call $p$ an order of the $A-$statistics. We conclude that like in the parastatistics all Fock spaces are labelled with positive integers $p$, the order of the statistics.

The equation (72) together with the commutation relations (44) of the $a-$operators determines completely the representation space of the creation and annihilation operators of order $p$. The $A-$statistics can be defined by the relations (44). The representations of the statistics can be obtained from (72). In this case all calculations can be done without using any Lie algebraical properties of the $a-$operators. Clearly this point of view is convenient for generalization to the case of infinite and in particular to continuum number of operators. The Lie algebraical structure however helps a lot in all calculations. Therefore we shall continue to consider a finite number of pairs $a_1^{\pm}, \ldots, a_n^{\pm}$ of $a-$operators and on a later stage we shall let $n \to \infty$.

Let us consider some Lie-algebraical properties of the Fock spaces. In the $A_n-$module $W$ with a highest weight $\Lambda = (L_0, L_1, \ldots, L_n)$ an arbitrary weight $\lambda = (l_0, l_1, \ldots, l_n)$ can be represented as

$$\lambda = \Lambda + \sum_i k_i \omega_i, \quad k_i \in N_0,$$

where

$$\omega_i \in \Sigma^- = \{h^i - h^j | i > j = 0, 1, \ldots, n\}.$$

Since the sum of the first $m$ co-ordinates, $m = 1, 2, \ldots, n$ of an arbitrary negative root is non-positive this is true also for the vector $\sum_i k_i \omega_i$ with $k_i$ non-negative integers. Therefore for an arbitrary weight $\lambda$ we have

$$l_0 + l_1 + \ldots + l_m \leq L_0 + L_1 + \ldots + L_m, \quad m = 0, 1, \ldots, n.$$

From this inequality and the circumstance that the weight system is invariant under the permutation of the orthogonal co-ordinates we conclude that the vector $\lambda = (l_0, l_1, \ldots, l_n)$ with integer co-ordinates is a weight if

$$l_{i_0} + l_{i_1} + \ldots + l_{i_m} \leq L_0 + L_1 + \ldots + L_m \tag{73}$$

where $i_0 \neq i_1 \neq \ldots, \neq i_m = 0, 1, \ldots, n; \;\; m = 0, 1, \ldots, n$. Clearly (73) is equality for $m = n$.

*Lemma 3.* All weights of the $A_n-$module of Fock $W_p$ with order of the statistics $p$ are simple.

*Proof.* An arbitrary weight vector $x_\lambda \in W_p$ with a weight $\lambda$ is generated from $x_\Lambda$ with polynomials of the creation operators,

$$x_\lambda = P(a_1^+, \ldots, a_n^+) x_\Lambda. \tag{74}$$

Therefore the weight $\lambda = (l_0, l_1, \ldots, l_n)$ of $x_\lambda$ can be represented as

$$\lambda = \Lambda + \sum_{i=1}^{n} k_i(-l^0 + h^i), \quad k_i \in N_0. \tag{75}$$

In terms of the co-ordinates the last relation reads as

$$(l_0, l_1, \ldots, l_n) = (p, 0, \ldots, 0) + (-\sum_{i=1}^{n} k_i, k_1, k_2, \ldots, k_n). \tag{76}$$

Hence $k_i = l_i, \; i = 1, \ldots, n$ and an arbitrary weight $\lambda$ is represented uniquely in the form (75). In terms of the weight vectors this gives that $P(a_1^+, \ldots, a_n^+)$ in (74) is homogeneous with respect to every creation operator $a_i^+$:

$$P(a_1^+, \ldots, \alpha a_i^+, \ldots, a_n^+) = \alpha^{l_i} P(a_1^+, \ldots, a_i^+, \ldots, a_n^+).$$

Since the creation operators commute,

$$P(a_1^+, a_2^+, \ldots, a_n^+) = (a_1)^{l_1}(a_2)^{l_2} \ldots (a_n)^{l_n}.$$

Therefore every vector $x_\lambda$ with a weight $\lambda = (l_0, l_1, \ldots, l_n)$ is collinear to the vector

$$(a_1^+)^{l_1}(a_2^+)^{l_2} \ldots (a_n^+)^{l_n} x_\Lambda$$

and the corresponding weight space is one-dimensional.

This lemma has no analogy in the parastatistics. For instance the states $b_i^+ b_j^+ |0\rangle$ and $b_j^+ b_i^+ |0\rangle,\ i \neq j$ have one and the same weight but in general are linearly independent.

In the following lemma we prove one important property of the $A-$statistics.

*Lemma 4.* Given $A_n-$module of Fock $W_p$ with an order of the statistics $p$. The vector

$$(a_1^+)^{l_1}(a_2^+)^{l_2} \ldots (a_n^+)^{l_n} |0\rangle \qquad (76')$$

is not zero if and only if

$$l_1 + l_2 + \ldots + l_n \leq p. \qquad (77)$$

In particular in the Fock space $W_p$ there can be no more than $p$ particles.

*Proof.* In the previous lemma we saw that the vector (76') has a weight

$$\lambda = (p - l_1 - \ldots - l_n, l_1, \ldots, l_n). \qquad (78)$$

If $l_1 + \ldots + l_n \leq p$, then clearly (73) holds because $L_0 + \ldots + L_m = p,\ m = 0, 1, \ldots, n$. Therefore $\lambda$ is a weight. There should exists at least one weight vector with weight $\lambda$. Since the multiplicity of $\lambda$ is one, this is the vector (76') and hence this vector is not zero.

If $l_1 + \ldots + l_n > 0$, the weight (78) does not fulfil the inequality (73) for $m = n - 1$ and $l_{i_0} = l_1,\ l_{i_1} = l_2,\ l_{i_{n-1}} = l_n$ and the corresponding weight vector (76') is zero.

From (76') and (78) we conclude

$$h_i (a_1^+)^{l_1}(a_2^+)^{l_2} \ldots (a_n^+)^{l_n} |0\rangle = l_i (a_1^+)^{l_1}(a_2^+)^{l_2} \ldots (a_n^+)^{l_n} |0\rangle, \quad i = 1, \ldots, n. \qquad (79)$$

The operator $h_i$ is the number operator $N_i$ of the particles in the state $i$. The number operator $N$ is

$$N = N_1 + N_2 + \ldots + N_n. \qquad (80)$$

We obtain

$$N (a_1^+)^{l_1}(a_2^+)^{l_2} \ldots (a_n^+)^{l_n} |0\rangle = (l_1 + \ldots + l_n)(a_1^+)^{l_1}(a_2^+)^{l_2} \ldots (a_n^+)^{l_n} |0\rangle. \qquad (81)$$

## 4. Matrix elements of the creation and annihilation operators

The numbers $l_1, \ldots, l_n$ together with the order of the $A-$statistics $p$ determine uniquely the state (76'). We introduce the notation

$$|p; l_1, l_2, \ldots, l_n\rangle = (a_1^+)^{l_1}(a_2^+)^{l_2}\ldots(a_n^+)^{l_n}|0\rangle. \tag{82}$$

The set of all vectors (82) constitute a basis of weight vectors in the Fock space $W_p$. The correspondence between the weight vectors and the weights written in their orthogonal co-ordinates reads as

$$|p; l_1, l_2, \ldots, l_n\rangle \quad \leftrightarrow \quad (p - \sum_{i=1}^{n} l_i, l_1, l_2, \ldots, l_n). \tag{83}$$

One has to remember that the notation $|p; l_1, l_2, \ldots, l_n\rangle$ is defined for $l_1 + \ldots + l_n \leq p$.

We now proceed to calculate the matrix elements on $n$ pairs of creation and annihilation operators $a_1^\pm, \ldots, a_n^\pm$ in the $A_n-$module of Fock $W_p$ with order of the statistics $p$.

We can write immediately

$$\begin{aligned} h_0|p; l_1, l_2, \ldots, l_n\rangle &= (p - \sum_{i=1}^{n} l_i)|p; l_1, l_2, \ldots, l_n\rangle, \\ h_i|p; l_1, l_2, \ldots, l_n\rangle &= l_i|p; l_i l_1, l_2, \ldots, l_n\rangle, \quad i = 1, \ldots, n. \end{aligned} \tag{84}$$

These equations follow from the observation that the orthogonal co-ordinates of the weight (83) are eigenvalues of the operators (28) on the weight vector (82). Since

$$[a_i^-, a_i^+] = h_0 - h_i$$

we have

$$[a_i^-, a_i^+]|p; l_1, l_2, \ldots, l_n\rangle = (p - L - l_i)|p; l_1, l_2, \ldots, l_n\rangle, \tag{85}$$

where $L = l_1 + l_2 + \ldots + l_n$.

First we calculate the matrix elements of $a_1^-$.

$$\begin{aligned} a_1^-|p; l_1, l_2, \ldots, l_n\rangle &= [a_1^-, (a_1^+)^{l_1}(a_2^+)^{l_2}\ldots(a_n^+)^{l_n}]|0\rangle \\ &= [a_1^-, (a_1^+)^{l_1}](a_2^+)^{l_2}\ldots(a_n^+)^{l_n}|0\rangle + (a_1^+)^{l_1}a_1^-(a_2^+)^{l_2}\ldots(a_n^+)^{l_n}|0\rangle. \end{aligned} \tag{86}$$

The second term in the last equality vanishes. Indeed the vector

$$a_1^-(a_2^+)^{l_2}\ldots(a_n^+)^{l_n}|0\rangle$$

would have had a weight

$$(p - \sum_{i=2}^{n} +1, -1, l_2, l_3, \ldots, l_n)$$

which is impossible since $l_0 + l_2 + l_3 + \ldots + l_n = p + 1 > p$.

Using (84), for the first term we obtain

$$\sum_{i=0}^{l_1-1}(a_1^+)^i[a_i^-, a_1^+](a_1^+)^{l_1-i-1}(a_2^+)^{l_2}\ldots(a_n^+)^{l_n}|0\rangle =$$
$$\sum_{i=0}^{l_1-1}(p - L - l_1 + 2i + 2)(a_1^+)^{l_1-1}(a_2^+)^{l_2}\ldots(a_n^+)^{l_n}|0\rangle.$$

This gives

$$a_1^-|p;l_1,l_2,\ldots,l_n\rangle = l_1(p-\sum_{i=1}^{n} l_i+1)|p;l_1-1,l_2,\ldots,l_n\rangle.$$

The generalization for $a_i^-$ is evident:

$$a_i^-|p;l_1,\ldots l_{i-1},l_i,l_{i+1},\ldots,l_n\rangle = l_i(p-\sum_{k=1}^{n} l_k+1)|p;l_1\ldots,l_{i-1},l_i-1,l_{i+1}\ldots,l_n\rangle. \tag{87}$$

Moreover

$$a_i^+|p;l_1,\ldots l_{i-1},l_i,l_{i+1},\ldots,l_n\rangle = |p;l_1\ldots,l_{i-1},l_i+1,l_{i+1}\ldots,l_n\rangle. \tag{88}$$

The metric in $W_p$ is defined in a complete analogy with the scalar product in the Fock space of Bose (or Fermi) operators.

Postulate

$$\begin{aligned}
&a)\quad \langle 0|0\rangle = 1,\\
&b)\quad \langle 0|a_i^+ = 0,\quad i=1,\ldots,n,\\
&c)\quad ((a_1^+)^{m_1}(a_2^+)^{m_2}\ldots(a_n^+)^{m_n}|0\rangle,(a_1^+)^{l_1}(a_2^+)^{l_2}\ldots(a_n^+)^{l_n}|0\rangle) =\\
&\qquad = \langle 0|(a_1^-)^{m_1}(a_2^-)^{m_2}\ldots(a_n^-)^{m_n}(a_1^+)^{l_1}(a_2^+)^{l_2}\ldots(a_n^+)^{l_n}|0\rangle).
\end{aligned} \tag{89}$$

The vectors $|p;l_1,\ldots,l_n\rangle$ constitute an orthogonal basis in $W_p$. To show this, suppose that in (89) some $m_i\neq l_i$ and let $m_i>l_i$. Then the vector

$$(a_i^-)^{m_i}(a_1^+)^{l_1}\ldots(a_i^+)^{l_i}\ldots(a_n^+)^{l_n}|0\rangle) = 0$$

since otherwise there has to exists a weight

$$(p-\sum_{j=1}^{n} l_j+m_i,l_1,\ldots,l_{i-1},-(m_i-l_i),l_{i+1},\ldots,l_n)$$

which is impossible. For $m_i<l_i$ the same result can be obtained from the hermitian conjugate of (89). If $m_i=l_i,\ i=1,\ldots,n$ we obtain

$$(|p;l_1,\ldots,l_n\rangle,|p;l_1,\ldots,l_n\rangle) = \frac{p!}{(p-L)!}\prod_{i=1}^{n} l_i!, \tag{90}$$

where $L=l_1+\ldots+l_n$.

As an orthogonal basis in $W_p$ one can accept the vectors

$$|p;l_1,\ldots,l_n) = \sqrt{\frac{(p-L)!}{p!}}\frac{(a_1^+)^{l_1}\ldots(a_n^+)^{l_n}}{\sqrt{l_1!l_2!\ldots l_n!}}|0\rangle. \tag{91}$$

In this basis we have for the matrix elements

$$a_i^+|p;l_1,\ldots l_{i-1},l_i,l_{i+1},\ldots,l_n) = \sqrt{(l_i+1)(p-\sum_{j=1}^{n} l_j)}|p;l_1\ldots,l_{i-1},l_i+1,l_{i+1}\ldots,l_n). \tag{92}$$

$$a_i^-|p;l_1,\ldots l_{i-1},l_i,l_{i+1},\ldots,l_n) = \sqrt{l_i(p-\sum_{j=1}^{n} l_j+1)}|p;l_1\ldots,l_{i-1},l_i-1,l_{i+1}\ldots,l_n). \tag{93}$$

The matrix elements of the $a-$operators do not depend on $n$. Therefore the result can be extended in an evident way to the case of infinite number of operators.

Finally, we point out one interesting property of the $A-$statistics. Introduce the operators

$$A_i^{\pm} = \frac{a_i^{\pm}}{\sqrt{p}}, \quad i = 1, \ldots, n \tag{94}$$

and consider the matrix elements of these operators on states with number of particles much less then $p$,

$$l_1 + l_2 + \ldots + l_n << p. \tag{95}$$

From (92-93) we obtain

$$\begin{aligned} a_i^-|p; l_1, \ldots l_{i-1}, l_i, l_{i+1}, \ldots, l_n\rangle &\simeq \sqrt{l_i}\, |p; l_1 \ldots, l_{i-1}, l_i - 1, l_{i+1} \ldots, l_n\rangle, \\ a_i^+|p; l_1, \ldots l_{i-1}, l_i, l_{i+1}, \ldots, l_n\rangle &\simeq \sqrt{l_i + 1}\, |p; l_1 \ldots, l_{i-1}, l_i + 1, l_{i+1} \ldots, l_n\rangle. \end{aligned} \tag{96}$$

In a first approximation

$$\begin{aligned} &[A_i^+, A_j^+] = [A_i^-, A_j^-], \quad \text{exact commutators,} \\ &[A_i^-, A_j^+] = \delta_{ij}, \quad \text{if } l_1 + l_2 + \ldots + l_n << p. \end{aligned} \tag{97}$$

Moreover if (95) holds, then

$$|p; l_1, \ldots, l_n\rangle = \frac{(A_1^+)^{l_1} \ldots (A_n^+)^{l_n}}{\sqrt{l_1! l_2! \ldots l_n!}} |0\rangle. \tag{98}$$

We see that if the $A-$statistics allows a large number of particles $p$, then the commutation relations of the operators $A_i^{\pm}$ on states with $l_1 + l_2 + \ldots + l_n << p$ coincide in a first approximation with the Bose creation and annihilation operators. In the limit $p \to \infty$ the operators $A_i^{\pm}$ reduce to Bose operators.

This property has also an interesting Lie-algebraical consequence. It shows that the limits of certain representations (the Fock representations) of the simple algebra $A_n$ leads to a representation of the solvable Lie algebra of Bose operators.

We have considered the statistics corresponding to the algebra of the unimodular group. In a similar way one can introduce a concept of $C-$ and $D-$statistics [9] or of statistics that correspond to other semisimple Lie algebras [5].

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